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STABILITY OF OSCILLATORY TWO PHASE COUETTE FLOW

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ABSTRACT

We investigate the stability of two phase Couette flow of different liquids bounded between plane parallel plates. One of the plates has a time dependent velocity in its own plane, which is composed of a constant steady part and a time harmonic component. In the absence of time harmonic modulations the flow can be unstable to an interfacial instability if the viscosities are different and the more viscous fluid occupies the thinner of the two layers. Using Floquet theory, we show analytically in the limit of long waves, that time periodic modulations in the basic flow can have a significant influence on flow stability. In particular, flows which are otherwise unstable for extensive ranges of viscosity ratios, can be stabilized completely by the inclusion of background modulations, a finding that can have useful consequences in many practical applications.

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1 Introduction

It is well known that plane Couette flow of two superposed fluids of different viscosities can be unstable. This instability is absent when the viscosities are equal and is a result of an interfacial deflection growing due to viscosity stratification. This mechanism was first described by Yih (1967), who used perturbation theory to obtain an analytic expression for the growth rate in the long wavelength limit. The results can be summarized as follows (for clarity, in the absence of gravity): Instability is possible only if the viscosities are different. Arrangements where the more viscous fluid occupies a thinner layer than the less viscous one are unstable, whereas converse arrangements are linearly stable to long waves. The instability is present at all Reynolds numbers, although in the limit of zero Reynolds number the growth rates become asymptotically small. Flow, therefore, is necessary to excite growing waves. Moderate surface tension is negligible to leading order for long waves, but acts to stabilize very short waves. Yih's work was extended more recently by other investigators. Hooper and Boyd (1983) considered the linear stability of two co-flowing viscous liquids of different viscosities separated by an interface and extending to infinity. This is Yih's problem in the absence of walls and can be used as a useful model in the study of short waves (short waves at the interface would not feel the effect of the walls, to leading order at least). Instability is found in all cases. Long waves are stable for all viscosity ratios, while short waves, in the absence of surface tension are unstable with asymptotically small growth rates - when included, surface tension damps out all waves which are short enough. Disturbances with general wavelengths are unstable and a maximum growth rate is attained at some viscosity ratio. This work was extended by Hooper (1985) who included a bounding moving wall for the lower fluid. The flow is shown to be unstable to long waves as long as the bounded layer is also the more viscous one, in agreement with Yih's findings. An explicit expression for the eigenvalues is also given in terms of Airy functions.

The stability of Yih's problem for general disturbances and including density differences and surface tension, was first solved numerically by Renardy (1985). She presents specific examples that show that the general long wavelength result described by Yih and Hooper and Boyd, can be extended to interfacial waves of arbitrary wavelengths. Another interesting feature is the finding of a second mode of instability at relatively high Reynolds numbers; this second mode has long wavelength and growth rates comparable to the first interfacial mode. Renardy (1987) carried out an analytical study of the same problem but for fluids with slightly different mechanical properties. Analytical expressions for the eigenvalues are constructed and instability is established in the thin layer limit if the thin layer is occupied by the more viscous fluid.

The stability characteristics of oscillatory flows are much more complicated than those of their steady counterparts. There are many examples where the inclusion of an oscillatory component to the steady flow can enhance or reduce stability (for plane Poiseuille flow, for example, see Grosch and Salwen (1968), Hall (1975), von Kerczek (1982); for a review on the stability of oscillatory flows see Davis (1975)). As far as we know the stability of oscillatory two-phase viscous flow has not been studied. Yih (1968) considered the stability of a viscous fluid layer on a flat plate performing a simple harmonic motion in its own plane. There is no viscosity stratification (the upper fluid is air) and the flow in the absence of the oscillation contains no steady velocity component and is linearly stable. Yih showed, using a long wave Floquet theory, that the oscillatory flow can become unstable even though in the absence of oscillations the flow is stable. This was extended by von Kerczek (1987) to flow down a vertical plate which is performing a simple harmonic motion in its own plane. This flow is unstable even in the absence of oscillations and von Kerczek uses Yih's long wavelength expansion to establish windows of instability. Our interest is to apply such a study to an oscillatory two-phase flow. In the problem we study here, depending on the fluid occupation areas in the unperturbed state (see earlier comments), the flow can be stable or unstable. In the former case, therefore, a quasi-static approach would yield background velocity profiles for each instant in time (time is treated as a parameter in quasi-steady theories) which are linearly stable while in the latter instance instability ensues for all parametrized profiles. A more yielding approach is that of Floquet theory where stability or instability is judged on the overall growth or decay with time of a perturbation over a complete period of the forced oscillation. Analytically this means that time-periodic solutions are constructed which can amplify, remain neutral or decay depending on whether the Floquet exponent is positive, zero or negative. We present representative results for several cases which in the absence of background oscillations the flow is unstable while imposed oscillations can stabilize the flow completely. On the other hand, flows which are stable can become unstable. We note that all the results given here are in terms of explicit but long formulae which yield stability results with little computational effort. Such explicit solutions are of considerable value in the generalization of the stability problem to arbitrary wavenumbers by use of continuation methods for instance.

The article is organized as follows. Section 2 derives the undisturbed flow as an exact shear flow solution of the Navier-Stokes equations. In Section 3 the linear stability problem is formulated and the interfacial conditions are written down. Explicit solutions are developed by an expansion procedure carried out to three orders, so that we may determine the first stage where the expansion of the Floquet exponent is a real term. In Section 4 we present the results of the stability analysis and finally in Section 5 we draw some conclusions.

2 The basic state

Two incompressible fluids of equal densities, ρ, and different viscosities, occupy a region of depth L between parallel infinite flat plates. The lower plate is fixed and the upper plate moves in its plane, with a steady velocity U_0, together with superposed sinusoidal oscillations, so that its velocity at time t^* is given by $U_0 + \Lambda\cos(\omega t^*)$. The fluids are assumed to be immiscible and form separate layers, the upper fluid has viscosity μ_1, the lower fluid has viscosity μ_2 and a depth D. We denote these as regions (I) and (II) respectively. We look for exact solutions of the Navier-Stokes equations for a constant interface position and velocity component in the streamwise direction alone which depends on time and the vertical coordinate only. Using cartesian coordinates (x^*, y^*), the exact flow is described by the following equations

$$\begin{aligned} \frac{\partial \overline{U_1^*}}{\partial t^*} &= \frac{\mu_1}{\rho}\frac{\partial^2 \overline{U_1^*}}{\partial {y^*}^2}, \qquad && L \geq y^* \geq D \quad (I), \\ \frac{\partial \overline{U_2^*}}{\partial t^*} &= \frac{\mu_2}{\rho}\frac{\partial^2 \overline{U_2^*}}{\partial {y^*}^2}, \qquad && D \geq y^* \geq 0 \quad (II). \end{aligned}$$

The boundary conditions are, no slip at the walls,

$$\begin{aligned} \overline{U_1^*}(y^* = L) &= U_0 + \Lambda\cos(\omega t^*), \\ \overline{U_2^*}(y^* = 0) &= 0. \end{aligned}$$

and continuity of velocities and stresses at the interface,

$$\begin{aligned} \overline{U_1^*}(y^* = D) &= \overline{U_2^*}(y^* = D), \\ \mu_1 \frac{\partial \overline{U_1^*}}{\partial y^*}(y^* = D) &= \mu_2 \frac{\partial \overline{U_2^*}}{\partial y^*}(y^* = D). \end{aligned}$$

The solution is easily found by separation into a steady part and an oscillatory part of the same frequency as the plate motion. The basic flow is,

$$\begin{aligned} \overline{U_1^*} &= \frac{\mu_2 U_0 y^* - U_0 D(\mu_2 - \mu_1)}{L\mu_2 - D(\mu_2 - \mu_1)} + \Lambda\mathcal{R}\left\{\left[L_1 e^{(\beta^* y^*)} + L_2 e^{(-\beta^* y^*)}\right] e^{(i\omega t^*)}\right\}, \\ \overline{U_2^*} &= \frac{\mu_1 U_0 y^*}{L\mu_2 - D(\mu_2 - \mu_1)} + \Lambda\mathcal{R}\left\{2K \sinh\left(\frac{\beta^* y^*}{m^{\frac{1}{2}}}\right) e^{(i\omega t^*)}\right\}, \end{aligned}$$

where $\mathcal{R}$ denotes the real part, $\beta^* = (i\rho\omega/\mu_1)^{\frac{1}{2}}$ and we define the ratio of the lower fluid viscosity to that of the upper layer by the parameter $m = \mu_2/\mu_1$. The constants K, L_1 and L_2 are given by

$$K = \frac{1}{2\sinh\left(\frac{\beta^* D}{m^{\frac{1}{2}}}\right)\cosh(\beta^*(D - L)) + 2m^{\frac{1}{2}}\cosh\left(\frac{\beta^* D}{m^{\frac{1}{2}}}\right)\sinh(\beta^*(D - L))}, \qquad (1)$$

$$L_1 = K\exp(-\beta^* D)\left[m^{\frac{1}{2}}\cosh\left(\frac{\beta^* D}{m^{\frac{1}{2}}}\right) + \sinh\left(\frac{\beta^* D}{m^{\frac{1}{2}}}\right)\right], \tag{2}$$

$$L_2 = K\exp(\beta^* D)\left[\sinh\left(\frac{\beta^* D}{m^{\frac{1}{2}}}\right) - m^{\frac{1}{2}}\cosh\left(\frac{\beta^* D}{m^{\frac{1}{2}}}\right)\right]. \tag{3}$$

The flow is non-dimensionalized next with D as lengthscale and U_0 as velocity scale, while time t^* and pressure P^* are non-dimensionalized by D/U_0 and ρU_0^2 respectively. This introduces the following non-dimensional groups (used later also):

$$a = \frac{L}{D}, \qquad \Delta = \frac{\Lambda}{U_0}, \qquad \beta = \beta^* D,$$

$$R_e = \frac{U_0 D\rho}{\mu_1}, \qquad \gamma = \frac{\sigma R_e^2}{\rho D U_0^2}, \qquad \Omega = \frac{\omega D}{U_0}.$$

The parameter a measures the relative depths of the two fluids, Δ measures the magnitude of the externally imposed oscillations, β is a measure of the Stokes layer thickness due to the oscillatory flow, R_e is the Reynolds number, and Ω is the non-dimensional frequency of the plate oscillations. Under these scalings the non-dimensional no-slip condition becomes

$$\overline{U}_1(y=a) = 1 + \Delta\cos(\Omega t),$$

$$\overline{U}_2(y=0) = 0,$$

and the basic flow is:

$$\overline{U}_1 = \frac{my - m + 1}{am - m + 1} + \Delta\mathcal{R}\left\{\left[L_1 e^{(\beta y)} + L_2 e^{(-\beta y)}\right] e^{(i\Omega t)}\right\},$$

$$\overline{U}_2 = \frac{y}{am - m + 1} + \Delta\mathcal{R}\left\{2K\sinh\left(\frac{\beta y}{m^{\frac{1}{2}}}\right) e^{(i\Omega t)}\right\}.$$

The constants K, L_1 and L_2 appearing above are in their non-dimensional forms which are readily available from (1)-(3) by replacing $\beta^* D$ by β. When $\Delta = 0$ we recover the basic Couette flow of different fluids in a channel (see Yih (1967)).

3 Stability equations.

The equation governing the linear stability of parallel shear flows is the Orr-Sommerfeld equation and arises from linearization of the Navier-Stokes equations and a normal mode expansion. In the present problem the background flow is time-dependent and so the eigenfunctions depend both on t and y. In particular, the perturbation streamfunction is taken to have the form $\tilde{\phi}_{1,2}(x,y,t) = \phi_{1,2}(y,t)\,e^{i\alpha x}$ where α is the wavenumber of the disturbance and subscripts 1,2 denote quantities in regions (I) and (II) respectively. The stability equations

in regions (I) and (II) are, then,

$$\left(\frac{\partial}{\partial t}+i\alpha\overline{U}_1\right)\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)\phi_1-i\alpha\phi_1\frac{\partial^2\overline{U}_1}{\partial y^2}=\frac{1}{R_e}\left[\frac{\partial^4\phi_1}{\partial y^4}-2\alpha^2\frac{\partial^2\phi_1}{\partial y^2}+\alpha^4\phi_1\right], \tag{4a}$$

$$\left(\frac{\partial}{\partial t}+i\alpha\overline{U}_2\right)\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)\phi_2-i\alpha\phi_2\frac{\partial^2\overline{U}_2}{\partial y^2}=\frac{m}{R_e}\left[\frac{\partial^4\phi_2}{\partial y^4}-2\alpha^2\frac{\partial^2\phi_2}{\partial y^2}+\alpha^4\phi_2\right]. \tag{4b}$$

The linearized boundary conditions are

$$\phi_1(a)=0=\frac{\partial\phi_1}{\partial y}(a), \tag{5}$$

$$\phi_2(0)=0=\frac{\partial\phi_2}{\partial y}(0). \tag{6}$$

We now let the interface position be $y=1+\delta h(t)e^{i\alpha x}$ where $\delta << 1$ is the infinitesimally small amplitude of the perturbations. The linearized interfacial conditions are found by expansion of the exact conditions about $y=1$ in powers of δ and retention of the leading order contributions:

$$\phi_1 = \phi_2, \tag{7}$$

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y}+h\frac{(1-m)}{m}\frac{\partial\overline{U}_1}{\partial y}, \tag{8}$$

$$\frac{\partial^2\phi_1}{\partial y^2}+\alpha^2\phi_1 = m\left[\frac{\partial^2\phi_2}{\partial y^2}+\alpha^2\phi_2\right], \tag{9}$$

$$\begin{aligned}&\frac{\partial^3\phi_1}{\partial y^3}-3\alpha^2\frac{\partial\phi_1}{\partial y}-2\alpha^2h\frac{\partial\overline{U}_1}{\partial y} - R_e\left[\frac{\partial^2\phi_1}{\partial y\partial t}+i\alpha\overline{U}_1\frac{\partial\phi_1}{\partial y}-i\alpha\phi_1\frac{\partial\overline{U}_1}{\partial y}\right]\\ &= m\left(\frac{\partial^3\phi_2}{\partial y^3}-3\alpha^2\frac{\partial\phi_2}{\partial y}-2\alpha^2h\frac{\partial\overline{U}_2}{\partial y}\right) - R_e\left[\frac{\partial^2\phi_2}{\partial y\partial t}+i\alpha\overline{U}_2\frac{\partial\phi_2}{\partial y}-i\alpha\phi_2\frac{\partial\overline{U}_2}{\partial y}\right]\\ &\quad + \frac{i\alpha^3h\gamma}{R_e}.\end{aligned} \tag{10}$$

The linearized kinematic condition at $y=1$ reads

$$\frac{dh}{dt}+i\alpha h\overline{U}_1+i\alpha\phi_1=0. \tag{11}$$

Equations (4-11) constitute the partial differential system together with boundary conditions that governs the stability of the flow. The problem contains time and one space variable and in general requires integration in time as an initial boundary value problem. Stability or instability is determined by the large time evolution of initial perturbations corresponding to different wavenumbers.

In this work we proceed analytically by studying the stability characteristics of long waves ($\alpha << 1$). Such analysis provides a significant amount of information since it enables

development of closed form expressions which can be used to check whether unsteadiness in the background flow can affect the evolution of linear perturbations. In the Couette problem studied by Yih (1967), instability was established for all Reynolds numbers by an analogous long wave expansion. The analysis of long waves is also useful in guiding numerical solutions of the initial value problem at general wavenumbers.

3.1 Solution for long waves

We now proceed analytically by considering the asymptotic limit $\alpha \to 0$ which corresponds to disturbances with long streamwise wavelengths. We use a method similar to the Floquet theory applied by Yih (1968) to obtain analytically the growth-rate of the disturbance at the interface. We look for solutions for the streamfunction and interfacial amplitude as a power series in $\alpha \ll 1$ of the form, for $j = 1, 2$,

$$\begin{aligned} \phi_j(y,t) &= (\phi_{j0}(y,t) + \alpha\phi_{j1}(y,t) + \ldots)\exp((\theta_0 + \alpha\theta_1 + \ldots)t)\exp(ik\alpha t), \\ h(t) &= (h_0(t) + \alpha h_1(t) + \ldots)\exp((\theta_0 + \alpha\theta_1 + \ldots)t)\exp(ik\alpha t), \\ k &= k_0 + \alpha k_1 + \ldots \end{aligned}$$

where $k_0, k_1, \ldots, \theta_0, \theta_1, \ldots$ are real constants, and $h_0(t), h_1(t), \ldots$ are periodic in time t.

By writing the streamfunction and interfacial deflection in this form, the linear long wave stability of the unsteady flow can then be determined by calculating the first non-zero θ, which corresponds to exponential growth or decay of the disturbance to the basic flow.

We now consider the system of governing equations (4a-b) and the boundary and interface conditions (5-11) to leading order, $O(1)$. Firstly the kinematic condition, (11) yields

$$\frac{dh_0}{dt} = -\theta_0 h_0. \tag{12}$$

Now since θ_0 is real and h_0 is periodic we have two choices: we may take $h_0 = 0$ so that θ_0 is as yet unprescribed, or alternatively $\theta_0 = 0$ and then, without loss of generality, $h_0 = 1$. We shall in fact assume the latter, since $h_0 = 0$ leads to damped modes only as shown later in Section 3.2.

After making the transformation $\psi_{j0} = \phi_{j0} + \overline{U}_j$, for each region, $j = 1, 2$, the leading order system becomes

$$\frac{\partial^3\psi_{10}}{\partial y^2\partial t} = \frac{1}{R_e}\frac{\partial^4\psi_{10}}{\partial y^4}, \tag{13a}$$

$$\frac{\partial^3\psi_{20}}{\partial y^2\partial t} = \frac{m}{R_e}\frac{\partial^4\psi_{20}}{\partial y^4}, \tag{13b}$$

$$\psi_{10}(y=a,t) = 1 + \frac{\Delta}{2}\left[e^{i\Omega t} + c.c.\right], \tag{14a}$$

$$\psi_{20}(y=0,t) = 0, \tag{14b}$$

$$\frac{\partial\psi_{10}}{\partial y}(y=a,t) = \frac{m}{am-m+1} + \left[\frac{\beta\Delta}{2}\left(L_1 e^{\beta a} - L_2 e^{-\beta a}\right)e^{i\Omega t} + c.c.\right], \tag{14c}$$

$$\frac{\partial\psi_{20}}{\partial y}(y=0,t) = \frac{1}{am-m+1} + \left[\frac{\beta\Delta}{m^{\frac{1}{2}}} K e^{i\Omega t} + c.c.\right]. \tag{14d}$$

Here, $c.c.$ denotes the complex conjugate.

At the interface $y=1$ we have

$$\psi_{10} = \psi_{20}, \tag{15a}$$

$$\frac{\partial\psi_{10}}{\partial y} = \frac{\partial\psi_{20}}{\partial y}, \tag{15b}$$

$$\frac{\partial^2\psi_{10}}{\partial y^2} = m\frac{\partial^2\psi_{20}}{\partial y^2}, \tag{15c}$$

$$\frac{\partial^3\psi_{10}}{\partial y^3} - R_e\frac{\partial^2\psi_{10}}{\partial y\partial t} = m\frac{\partial^3\psi_{20}}{\partial y^3} - R_e\frac{\partial^2\psi_{20}}{\partial y\partial t}, \tag{15d}$$

These equations admit solutions of the form:

$$\begin{aligned}
\psi_{10} &= A_{10}(y-a)^3 + B_{10}(y-a)^2 + \frac{m(y-a)}{am-m+1} + 1 \\
&+ \left\{\left[C_{10}\sinh(\beta y) + D_{10}\cosh(\beta y) + E_{10}y + F_{10}\right]e^{i\Omega t} + \mathrm{c.c.}\right\}, \\
\psi_{20} &= A_{20}y^3 + B_{20}y^2 + \frac{y}{am-m+1} \\
&+ \left\{\left[C_{20}\sinh\left(\beta y m^{-\frac{1}{2}}\right) + D_{20}\cosh\left(\beta y m^{-\frac{1}{2}}\right) + E_{20}y + F_{20}\right]e^{i\Omega t} + c.c.\right\}.
\end{aligned}$$

The twelve constants are found analytically by substitution into the above boundary and interface conditions, the resulting equations which determine $A_{10}, A_{20}, \ldots, F_{10}, F_{20}$ are given in the appendix.

At the next order, $O(\alpha)$, the kinematic condition (11) becomes

$$\frac{dh_1}{dt} + \theta_1 + i\psi_{10}(1,t) + ik_1 = 0 \tag{16}$$

Our solution strategy in obtaining the eigenvalues analytically, is based in part on the perturbation eigenfunctions and interfacial amplitude being time periodic. In equation (16) above, $\psi_{10}(1,t)$ is the sum of a real constant and a time periodic function, while k_1, and θ_1 are real. In order to obtain periodic solutions $h_1(t)$ the following constraints need to be satisfied,

$$k_1 = -\psi_{10}^{(s)}(1) = A_{10}(a-1)^3 - B_{10}(1-a)^2 + \frac{m(a-1)}{am-m+1} - 1,$$

$$
\begin{aligned}
h_1 &= -\int i\psi_{10}^{(us)}(1,t)\,dt, \\
&= -\Omega^{-1}\left[C_{10}\sinh(\beta) + D_{10}\cosh(\beta) + E_{10} + F_{10}\right]e^{i\Omega t} - c.c.,
\end{aligned}
\tag{17}
$$

where the superscripts (s) and (us) denote the steady and unsteady parts respectively.

Before solving the $O(\alpha)$ system, let us first consider the $O(\alpha^2)$ kinematic condition, which can be written as

$$
\frac{dh_2}{dt} + \theta_2 + ik_1h_1 + ik_2 + ih_1\overline{U}_1 + i\phi_{11}(1,t) = 0.
$$

Using $\mathcal{I}$ to denote the imaginary part, we see that for $h_2(t)$ to be periodic in t we require,

$$
\theta_2 = \mathcal{I}\left\{\left[h_1\overline{U}_1 + \phi_{11}(1)\right]^{(s)}\right\}.
$$

Clearly then it is sufficient to solve for the steady part of the eigenfunction ϕ_{11} to $O(\alpha)$ in order to determine θ_2 and hence the linear stability of the interface. We note that at this order, products of functions which have a time dependence given by $\exp(\pm i\Omega t)$, yield additional steady terms in the governing equations and boundary/interface conditions. The eigenvalue θ_2 is therefore determined by both the steady and time oscillatory motion of the interface. The time independent perturbation equations are given by

$$
\frac{1}{R_e}\frac{\partial^4\phi_{11}^{(s)}}{\partial y^4} = i\left[\left(k_1+\overline{U}_1\right)\frac{\partial^2\phi_{10}}{\partial y^2} - \frac{\partial^2\overline{U}_1}{\partial y^2}\phi_{10}\right]^{(s)}, \tag{18a}
$$

$$
\frac{m}{R_e}\frac{\partial^4\phi_{21}^{(s)}}{\partial y^4} = i\left[\left(k_1+\overline{U}_2\right)\frac{\partial^2\phi_{20}}{\partial y^2} - \frac{\partial^2\overline{U}_2}{\partial y^2}\phi_{20}\right]^{(s)}. \tag{18b}
$$

They admit a particular solution

$$
\chi_{11} = iR_e\int_1^a\int_1^a\int_1^a\left[\left(k_1+\overline{U}_1\right)\frac{\partial\psi_{10}}{\partial y} - \psi_{10}\frac{\partial\overline{U}_1}{\partial y}\right]^{(s)}dydydy, \tag{19a}
$$

$$
\chi_{21} = im^{-1}R_e\int_0^1\int_0^1\int_0^1\left[\left(k_1+\overline{U}_2\right)\frac{\partial\psi_{20}}{\partial y} - \psi_{20}\frac{\partial\overline{U}_2}{\partial y}\right]^{(s)}dydydy, \tag{19b}
$$

so that the general solution of equations (18a-b) is

$$
\begin{aligned}
\phi_{11}^{(s)} &= A_{11}(y-1)^3 + B_{11}(y-1)^2 + C_{11}(y-1) + D_{11} + \chi_{11}, \\
\phi_{21}^{(s)} &= A_{21}(y-1)^3 + B_{21}(y-1)^2 + C_{21}(y-1) + D_{21} + \chi_{21}.
\end{aligned}
$$

The eight constants are found by applying the following boundary and interface conditions:

$$
\phi_{11}^{(s)}(y=a) = 0,
$$

$$
\begin{aligned}
\phi_{21}^{(s)}(y=0) &= 0, \\
\frac{\partial \phi_{11}^{(s)}}{\partial y}(y=a) &= 0, \\
\frac{\partial \phi_{21}^{(s)}}{\partial y}(y=0) &= 0, \\
\phi_{11}^{(s)} &= \phi_{21}^{(s)}, \\
\frac{\partial \phi_{11}^{(s)}}{\partial y} &= \frac{\partial \phi_{21}^{(s)}}{\partial y} - \frac{(m-1)}{m}\left[h_1 \frac{\partial \overline{U}_1}{\partial y}\right]^{(s)}, \\
\frac{\partial^2 \phi_{11}^{(s)}}{\partial y^2} &= m \frac{\partial^2 \phi_{21}^{(s)}}{\partial y^2}, \\
\frac{\partial^3 \phi_{11}^{(s)}}{\partial y^3} &- m \frac{\partial^3 \phi_{21}^{(s)}}{\partial y^3} - R_e \left[\frac{ik_1 \partial \phi_{10}}{\partial y} + i\overline{U}_1 \frac{\partial \phi_{10}}{\partial y} - i\phi_{10} \frac{\partial \overline{U}_1}{\partial y}\right]^{(s)} \\
&+ R_e \left[\frac{ik_1 \partial \phi_{20}}{\partial y} + i\overline{U}_2 \frac{\partial \phi_{20}}{\partial y} - i\phi_{20} \frac{\partial \overline{U}_2}{\partial y}\right]^{(s)}.
\end{aligned}
$$

The above solutions and boundary conditions provide a system of eight inhomogeneous algebraic equations for the eight constants $A_{11}, A_{21}, \ldots, D_{11}, D_{21}$ and are solved explicitly here, (see appendix).

3.2 Damped disturbances

Before presenting the results of the calculation of the eigenvalue θ_2, we show that the possibility $h_0 = 0$ yields damped waves. To achieve this we write the leading order streamfunction as

$$
\phi_{j0} = Q_j(y)\exp(\theta_0 t), \qquad \text{where } j = 1, 2.
$$

This satisfies the equations

$$
R_e \theta_0 \frac{d^2 Q_1}{dy^2} = \frac{d^4 Q_1}{dy^4}, \tag{20}
$$

$$
R_e \theta_0 \frac{d^2 Q_2}{dy^2} = m \frac{d^4 Q_2}{dy^4}, \tag{21}
$$

which are solved subject to

$$
\begin{aligned}
Q_1(0) = \frac{dQ_1}{dy}(0) &= 0, \\
Q_2(a) = \frac{dQ_2}{dy}(a) &= 0, \\
Q_1(1) &= Q_2(1),
\end{aligned}
$$

$$\begin{aligned}
\frac{dQ_1}{dy}(1) &= \frac{dQ_2}{dy}(1), \\
\frac{d^2Q_1}{dy^2}(1) &= m\frac{d^2Q_2}{dy^2}(1), \\
\frac{d^3Q_1}{dy^3}(1) - R_e\theta_0\frac{dQ_1}{dy}(1) &= m\frac{d^3Q_3}{dy^2}(1) - R_e\theta_0\frac{dQ_2}{dy}(1).
\end{aligned}$$

Multiplying (20) and (21) by $\frac{dQ_1}{dy}$ and $\frac{dQ_2}{dy}$ respectively, integrating between 0 and a, and imposing the above boundary and interface conditions yields

$$R_e\theta_0 = -m\frac{\left[\int_0^1 \left(\frac{d^2Q_1}{dy^2}\right)^2 dy + \int_1^a \left(\frac{d^2Q_2}{dy^2}\right)^2 dy\right]}{\left[\int_0^1 \left(\frac{dQ_1}{dy}\right)^2 dy + \int_1^a \left(\frac{dQ_2}{dy}\right)^2 dy\right]} < 0.$$

Hence all disturbances are damped for this case.

4 Results

The non-dimensional parameter Δ is the magnitude of the sinusoidal oscillations of the upper plate relative to the magnitude of the steady velocity. When $\Delta = 0$, this problem reduces to that of steady plane Couette flow of two superposed fluids of differing viscosity, (as characterized by the ratio m).

As an initial check on the validity of our analysis we firstly consider this steady case, which corresponds to the problem solved by Yih (1967). In order to make a direct comparison of the results we must first note that the length scale used by Yih (1967) is equivalent to $(L - D)$ whereas in this work we use the lower fluid depth D. Yih also defines a parameter $n = d_2/d_1$ the depth ratio of the lower to upper regions, a brief calculation shows that $n \equiv (a-1)^{-1}$. In view of these notational differences the Reynolds number and streamwise wavenumber are not the same as their counterparts defined in this work, we must first make the transformations $\alpha_{Yih} \to n\alpha$ and $R_{Yih} \to nR_e$, we hence write $J(m, a, \Delta = 0) = R_e^{-1}(a-1)^{-2}\theta_2$.

Calculation of θ_2, and hence J, is in theory an analytical task, in practice however, the twenty constants obtained in the integration of the $O(1)$ and $O(\alpha)$ systems, are lengthy expressions which are found most efficiently by use of a symbolic algebra package. For completeness, the simultaneous equations which determine these constants are given in the appendix.

For a given viscosity ratio m, depth ratio a, frequency Ω and magnitude Δ, the value of θ_2 is calculated. When the real Floquet exponent $\theta_2 > 0$ the disturbance to the basic flow will grow exponentially in time and the flow will be unstable, similarly negative θ_2 corresponds to a damped interfacial disturbance, and the flow is said to be linearly stable.

Figures 1(a) and 1(b) plot the values of J against m for a steady flow, $(\Delta = 0)$, when the lower fluid is more viscous than the upper layer. We have chosen equivalent depth ratios to those illustrated by Yih (1967), the results are identical.

Figure 1(a) corresponds to a flow in which the upper layer is deeper, for a more viscous lower layer the growth rate is positive and the flow is always unstable. With a shallower upper layer, Figure 1(b) indicates that there is a region of stability, depending on the size of the viscosity ratio, although for sufficiently large m the flow will become unstable.

For unequal depths of fluids the growth rate tends linearly to zero as m approaches unity, since this is a hidden mode for flows of equal viscosities.

To obtain results for a less viscous lower region, we have verified the following transformation given by Yih

$$mJ\left(m, n, \Delta = 0\right) = n^2 J\left(\frac{1}{m}, \frac{1}{n}, \Delta = 0\right), \qquad \text{where } n = (a-1)^{-1}. \tag{22}$$

The results for the steady flow when the more viscous fluid occupies the upper region can then be inferred from Figures 1(a) and 1(b).

Before discussing the results for the full time-oscillatory problem, let us first investigate this property in more detail. Yih observes that the phase speed of the disturbed flow, (c_0' using his notation), is equal in *magnitude* and *opposite in sign when the viscosities* and depths of the fluid layers are interchanged. This is because, the original flow may be recovered identically by making a Galilean transformation from the coordinates (x, y, t) to (ξ, y, t), where $\xi = x - t$, hence $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\right) \rightarrow \left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t} - \frac{\partial}{\partial \xi}\right)$. The upper boundary now becomes stationary, (relative to the moving frame of reference), and the lower plate moves with constant velocity -1. Since gravity is neglected for this problem, inverting the geometry leaves the eigenvalue, and hence the stability characteristics, unchanged. In order to compare the growth-rates quantitatively however we must note that now the Reynolds number, $R_e = U_0 D\rho/\mu_1$, and lengthscale D must be rescaled on the *new* upper fluid, hence the multiplying factor n^2/m, as given by equation (22) above.

We now consider the problem when $\Delta \neq 0$, the flow is dependent upon the time t and analogous conclusions from a change of reference frame do not hold and so for completeness we also need to consider arrangements when when the lower fluid is less viscous than the upper layer. Since the unsteady contribution is periodic, we need only consider $\Delta > 0$. In fact it is clearly seen that Δ is a multiplicative factor in the basic flow, ϕ_{10}, ϕ_{20} and also h_1, and these terms contribute to θ_2 as products, thus $-\Delta$ gives identical results.

When the lower fluid is more viscous, $m > 1$, the interface is destabilized by the introduction of an unsteady basic flow. Figure 2(a) shows that for $a = 3.5$ the already unstable steady mode, (broken line), is made more unstable upon increasing the magnitude of the

oscillations. Similarly, when the lower fluid is deeper, for example $a = 1.4$, so that in the absence of background time periodic modulations the flow is stable for moderate m at least, the interface is again destabilized as indicated by Figure 2(b). The value of m below which the flow is stable, decreases as Δ increases, for example when $\Delta = 0.2$, the neutral disturbance is obtained for a viscosity ratio $m \approx 86.2319$, whereas for $\Delta = 0.4$, $m \approx 66.2506$. This is shown more clearly by the neutral curve in Figure 3; if the magnitude of the oscillations increases beyond the critical value $\Delta \approx 0.5695$, the interface is unstable for all $m > 1$.

When the lower fluid is less viscous, $m < 1$, the results are more significant. For all depth ratios we find that the time-dependent oscillations dramatically stabilize the interfacial disturbance. For a deeper upper layer, for example $a = 2.25$ as in Figure 4(a), the range of viscosity ratios m, for which the real growth rate is negative is increased for the unsteady background flow and as Δ is increased the flow is stabilized further. Figure 4(b), illustrates how the unstable steady mode corresponding to a shallow upper layer, can be completely stabilized provided Δ is made large enough. Such behavior is shown collectively in Figure 5 which can be used to predict oscillation amplitudes which completely stabilize the flow for given depth ratios a. Figure 5 depicts the variation of neutral stability pairs m, Δ for three different depth ratios. In each case there is a global maximum oscillation amplitude above which the flow is linearly stable for all viscosity ratios. For example when the depth ratio is $a = 1.4$ we see that the flow is stable if $\Delta > 0.6412$ (approximately).

Finally we quantify the effect of varying the frequency of the oscillations, through the non-dimensional parameter Ω. Increasing the frequency reduces the effect of the unsteady terms and the stability of this fluid regime becomes comparable with that of steady plane Couette flow, (denoted by the broken line). These stability results can be understood by consideration of the unperturbed flow at high oscillation frequencies. In such a regime the flow separates into a Stokes layer in the vicinity of the oscillating wall, and away from this layer the flow is steady and corresponds to two phase Couette flow due to a boundary which moves with constant velocity. As long as the Stokes layer is thin compared to the thickness of the upper layer, therefore, the interfacial mode is expected to be insensitive to the wall modulations as indicated by our numerical results. The Stokes layer thickness, in nondimensional terms, is proportional to $R_e^{-1/2}\Omega^{-1/2}$, and for the highest values of Ω used in Figures 6(a)-(b) the ratio between the Stokes layer thickness and the distance of the unperturbed interface from the wall is approximately 0.025 and 0.25 respectively for a unit Reynolds number, confirming the arguments given above. On the other hand, as Ω is reduced the stabilizing or destabilizing tendencies characterized by $\Delta > 0$, are emphasized, as shown by Figures 6(a) and 6(b). It can be concluded, therefore, that from a practical point of view, stabilization due to time harmonic modulations is likely to occur as long as

the modulation frequency is not too large.

5 Conclusions

We have considered the effect of the inclusion of time harmonic modulations in the unperturbed velocity field of two phase Couette flow of different liquids. Our main aim is the demonstration of the stabilizing effect that such modulations can have on otherwise interfacially unstable flows, a finding that can have useful practical applications. To this end, we considered the unsteady partial differential stability system in the limit of long wavelength perturbations which can be solved exactly by use of Floquet and perturbation theory to yield analytical expressions for the first non-zero Floquet exponent which governs stability or instability. The main conclusion of this study is that inclusion of modulations with an amplitude larger than a certain parameter-dependent threshold, can completely stabilize flows which are unstable in the absence of modulations; for instance flows with the more viscous fluid occupying a thin layer and bounded by the moving wall can be completely stabilized for long waves, at least. At the same time, modulations can produce an adverse effect on the interfacial mode. Flow arrangements which are stable or unstable can become unstable or more unstable respectively. Finally, we emphasize that the conclusions of this work are valid for long wavelength perturbations which are useful in providing analytical solutions to a problem that needs to be addressed numerically in general. Further more, the physical conclusions reached for the long wavelength limit are expected to be indicative of the behavior of general wavelength time periodic perturbations.

Appendix

For the solution of the $O(1)$ system (13a-b), we impose boundary and interface conditions given by equations (14a-d) and (15a-d). These yield the following twelve equations, which determine the constants $A_{10}, A_{20}, \ldots, F_{10}, F_{20}$:

$$\begin{aligned}
A_{10} - mA_{20} &= 0, \\
3A_{10}(1-a) - 3mA_{20} + B_{10} - mB_{20} &= 0, \\
A_{10}(1-a)^3 - A_{20} + B_{10}(1-a)^2 - B_{20} &= 0, \\
3A_{10}(1-a)^2 - 3A_{20} + 2B_{10}(1-a) - 2B_{20} + \frac{m-1}{am-m+1} &= 0, \\
E_{10} - E_{20} &= 0, \\
F_{10} - F_{20} &= 0, \\
F_{10} + D_{20} &= 0, \\
C_{10}\sinh(\beta a) + D_{10}\cosh(\beta a) + E_{10}a + F_{10} - \frac{\Delta}{2} &= 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\beta\Delta}{2}\left[L_1 e^{(\beta a)} - L_2 e^{(-\beta a)}\right] &= \beta C_{10}\cosh(\beta a) + \beta D_{10}\sinh(\beta a) + E_{10}, \\
\beta m^{-\frac{1}{2}} C_{20} + E_{20} &= \beta m^{-\frac{1}{2}} \Delta K, \\
C_{10}\sinh(\beta) + D_{10}\cosh(\beta) &= C_{20}\sinh\left(\beta m^{-\frac{1}{2}}\right) + D_{20}\cosh\left(\beta m^{-\frac{1}{2}}\right), \\
C_{10}\cosh(\beta) + D_{10}\sinh(\beta) &= m^{-\frac{1}{2}}\left[C_{20}\cosh\left(\beta m^{-\frac{1}{2}}\right) + D_{20}\sinh\left(\beta m^{-\frac{1}{2}}\right)\right].
\end{aligned}$$

At $O(\alpha)$ the solution of (18a-b), requires a further eight equations, to determine the constants $A_{11}, A_{21}, \ldots, D_{11}, D_{21}$:

$$\begin{aligned}
A_{11} - mA_{21} &= 0 \\
A_{11}(a-1)^3 + B_{11}(a-1)^2 + C_{11}(a-1) + D_{11} + \chi_{11}(a) &= 0, \\
3A_{11}(a-1)^2 + 2B_{11}(a-1) + C_{11} + \frac{\partial \chi_{11}}{\partial y}(a) &= 0, \\
-A_{21} + B_{21} - C_{21} + D_{21} + \chi_{21}(0) &= 0 \\
3A_{21} - 2B_{21} + C_{21} + \frac{\partial \chi_{21}}{\partial y}(0) &= 0, \\
D_{11} - D_{21} + \chi_{11}(1) - \chi_{21}(1) &= 0, \\
C_{11} - C_{21} + \frac{\partial \chi_{11}}{\partial y}(1) - \frac{\partial \chi_{21}}{\partial y}(1) + \frac{(m-1)}{m}\left[h_1 \frac{\partial \overline{U}_1}{\partial y}\right]^{(s)} &= 0, \\
2B_{11} - 2mB_{21} + \frac{\partial^2 \chi_{11}}{\partial y^2}(1) - m\frac{\partial^2 \chi_{21}}{\partial y^2}(1) &= 0.
\end{aligned}$$

Where χ_{11} and χ_{21} are the particular solutions defined by equations (19a-b).

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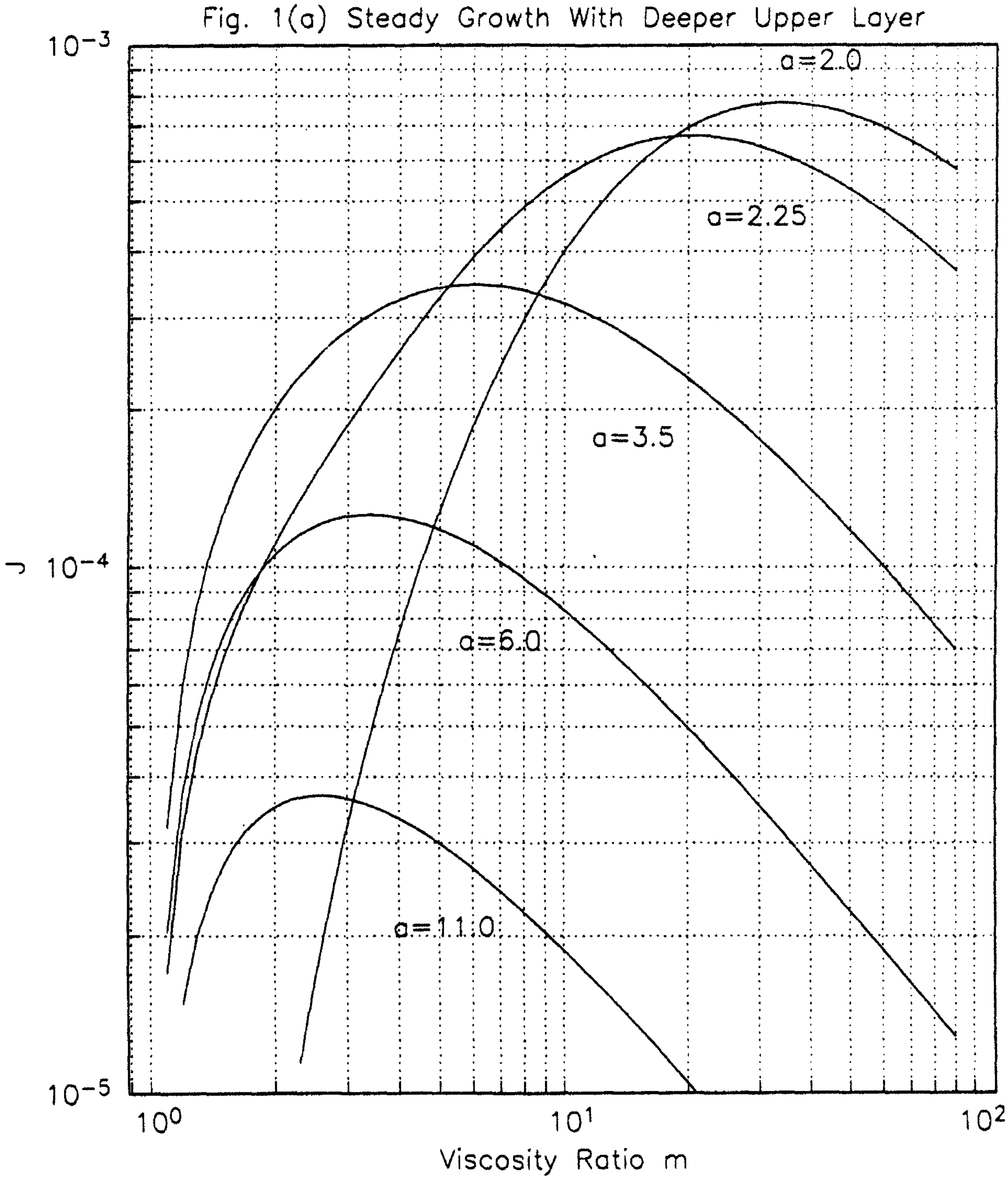
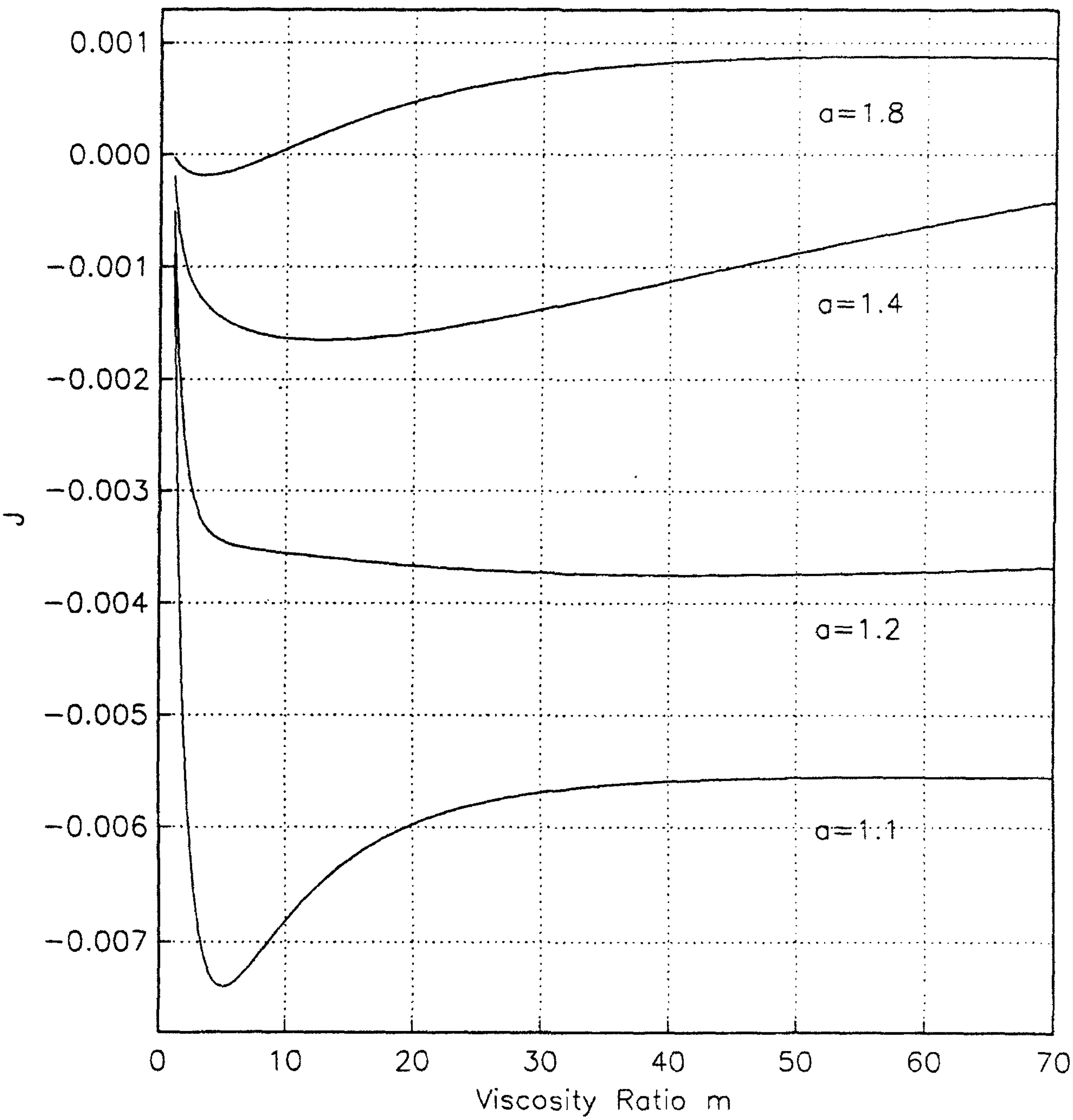


Fig. 1(a) Steady Growth With Deeper Upper Layer
J
10^{-3}
10^{-4}
10^{-5}
10^{0}
10^{1}
10^{2}
Viscosity Ratio m
a=2.0
a=2.25
a=3.5
a=6.0
a=11.0

Fig. 1(b) Steady Growth With Deeper Lower Layer



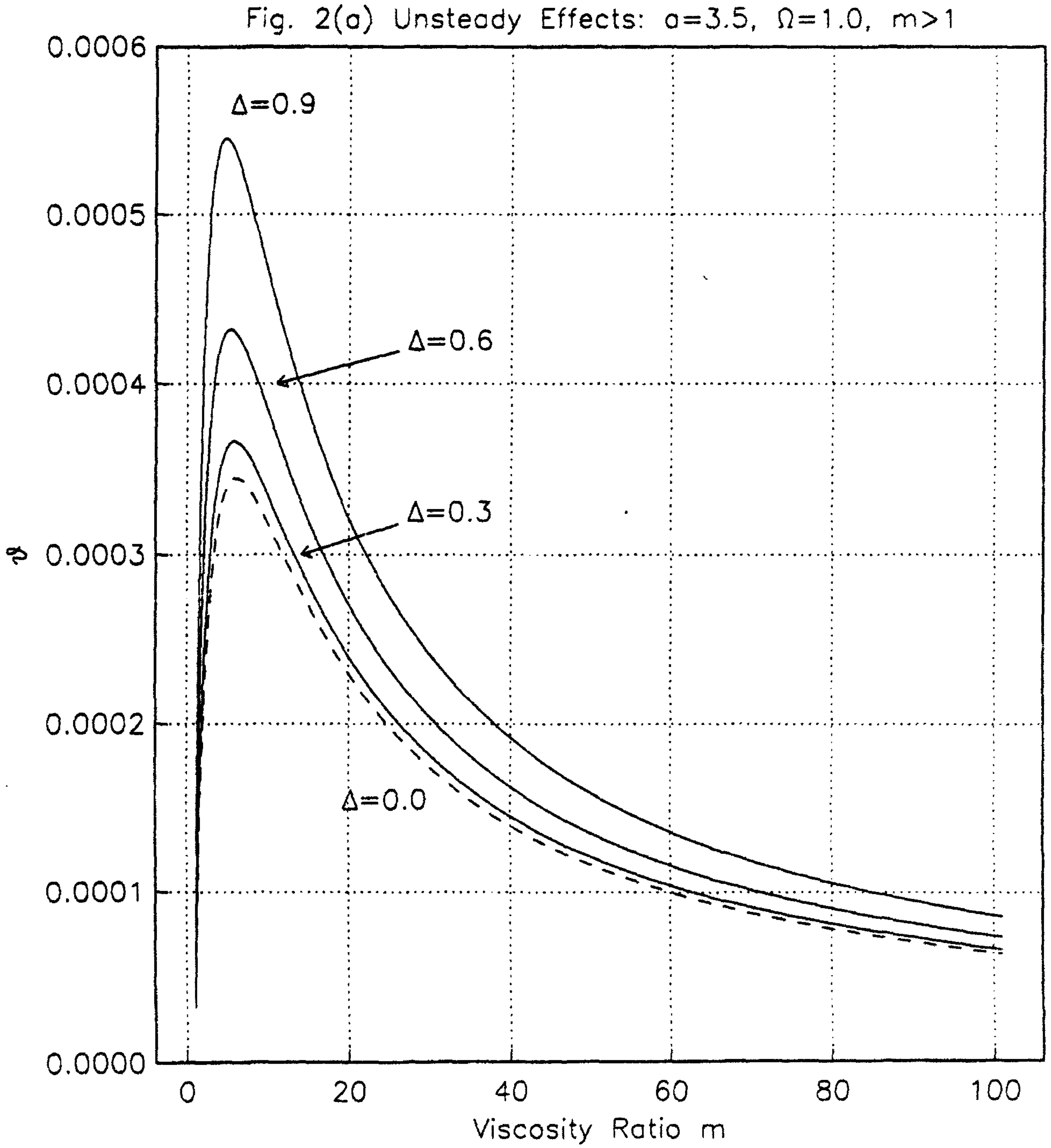
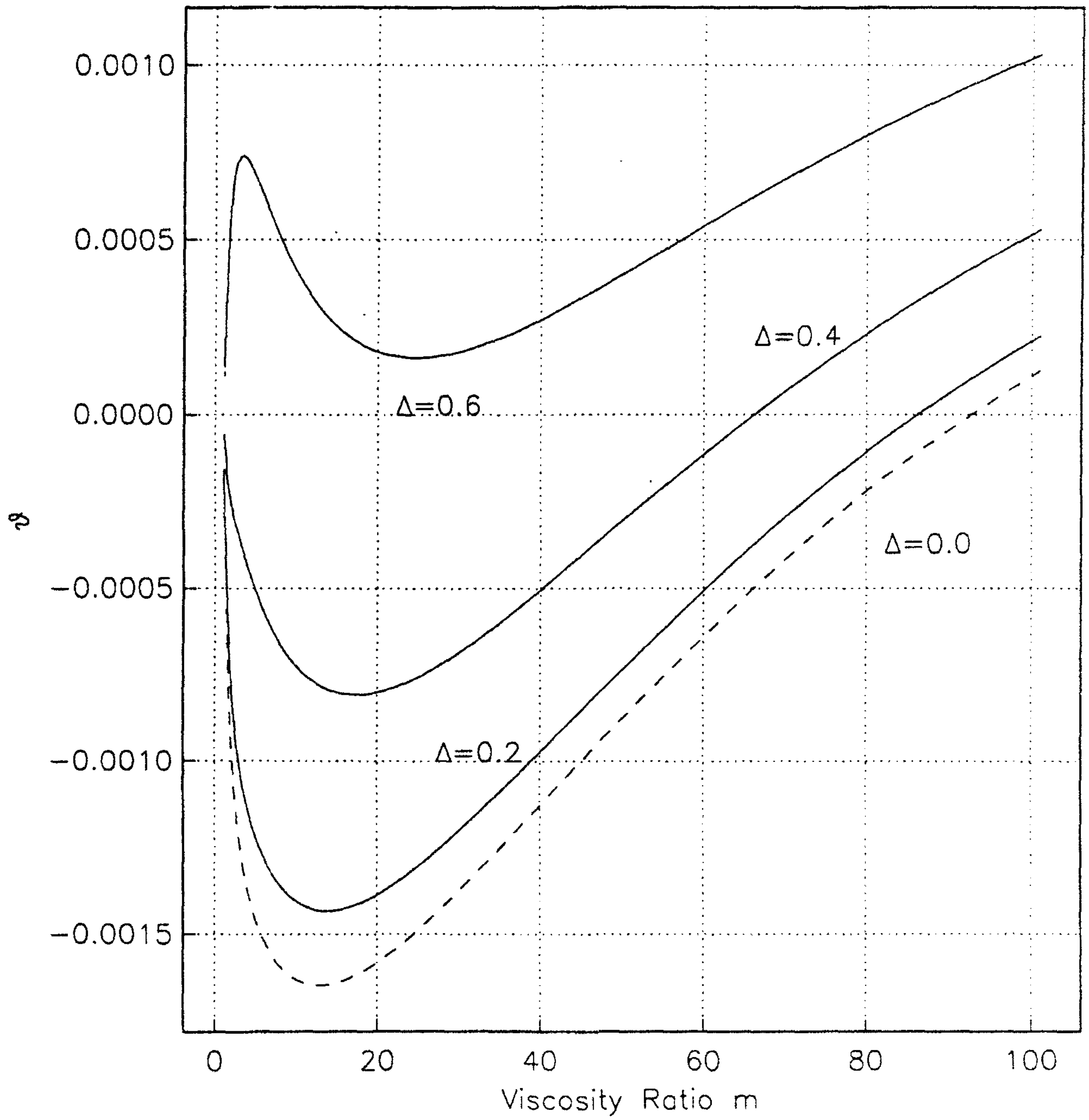


Fig. 2(a) Unsteady Effects: a=3.5, Ω=1.0, m>1
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0.0000
ϑ
Δ=0.9
Δ=0.6
Δ=0.3
Δ=0.0
0
20
40
60
80
100
Viscosity Ratio m

Fig. 2(b) Unsteady Effects: $a=1.4$, $\Omega=1.0$, $m>1$



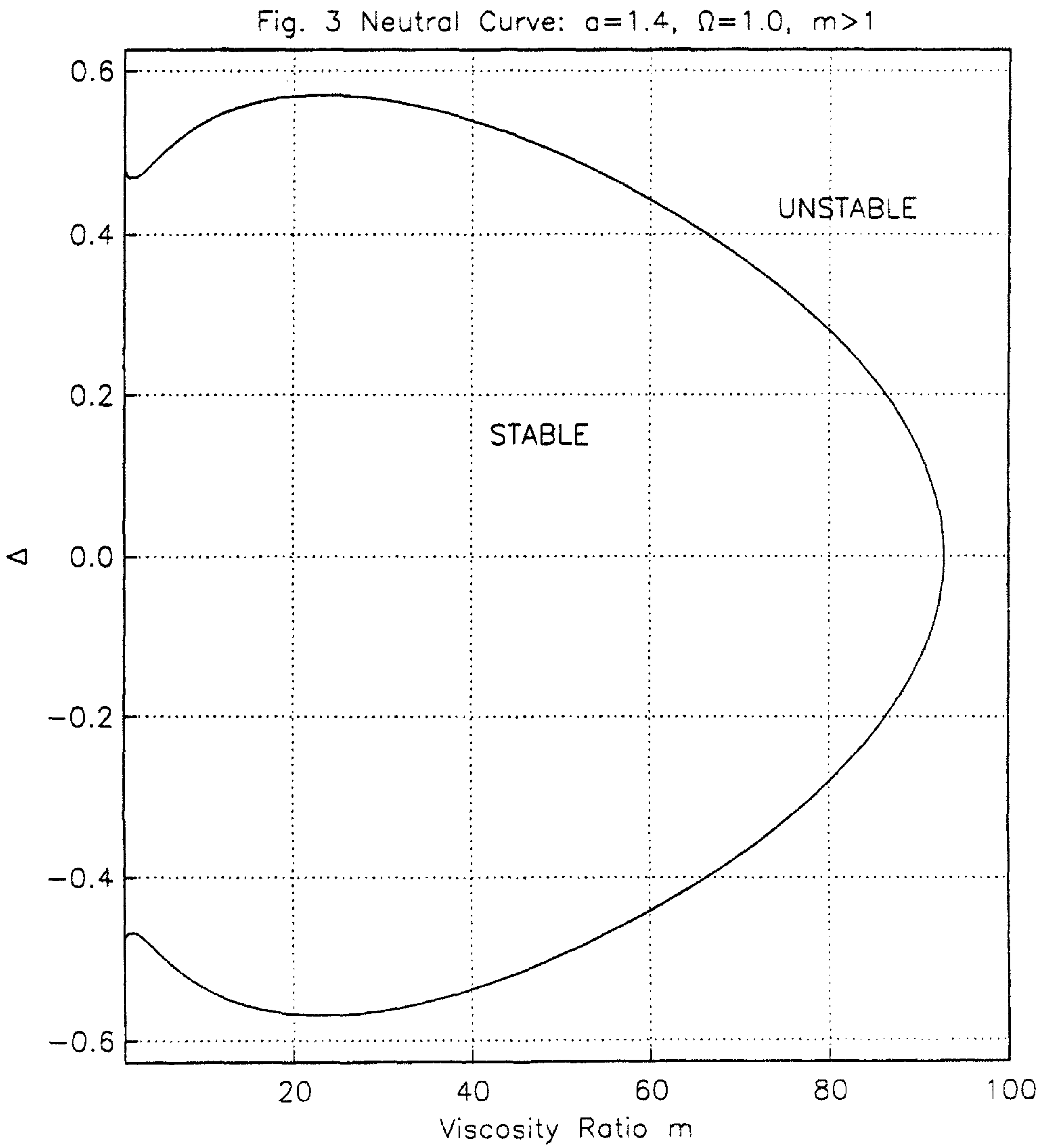


Fig. 3 Neutral Curve: a=1.4, Ω=1.0, m>1
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
Δ
UNSTABLE
STABLE
20
40
60
80
100
Viscosity Ratio m

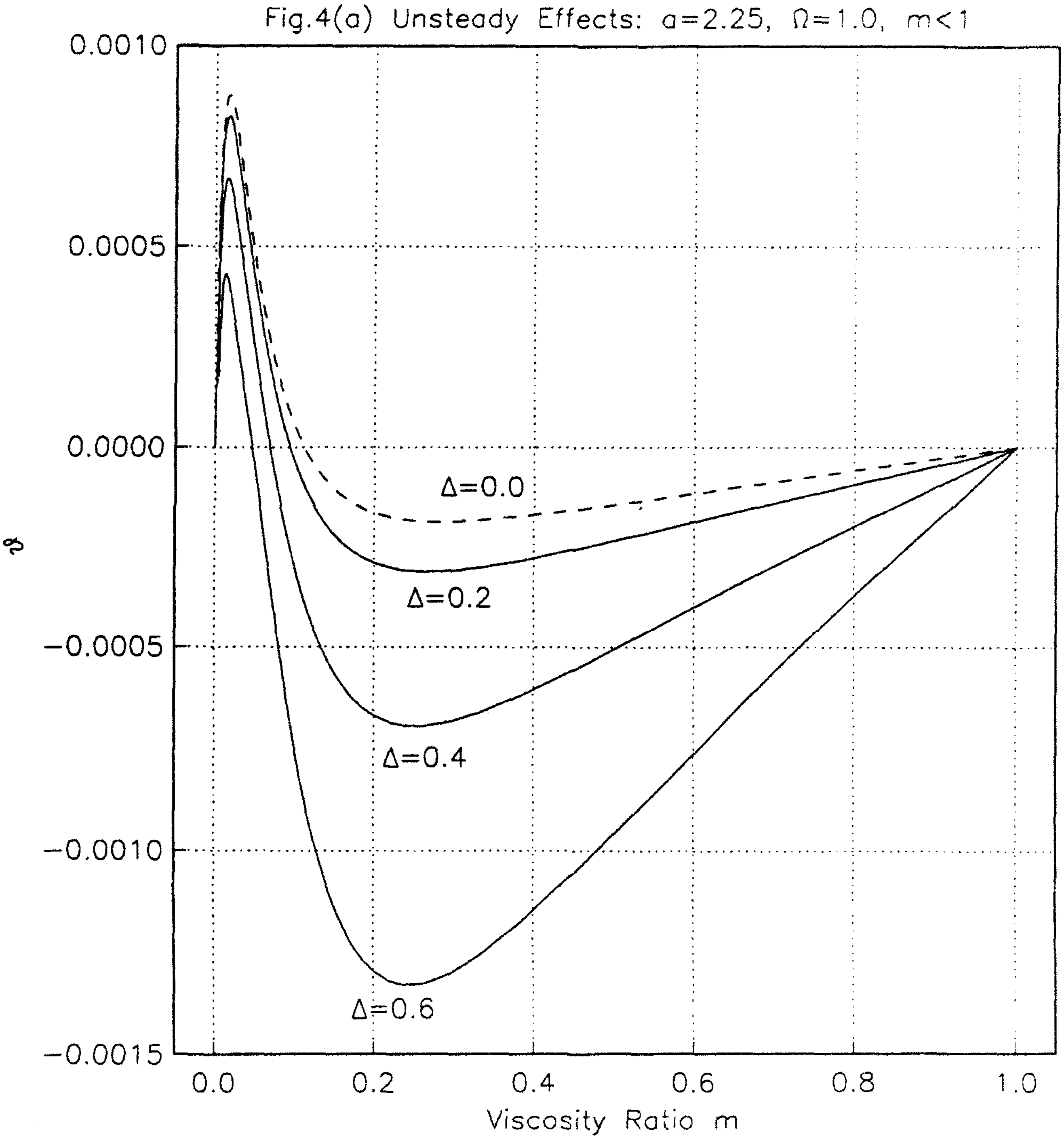


Fig.4(a) Unsteady Effects: a=2.25, Ω=1.0, m<1
0.0010
0.0005
0.0000
−0.0005
−0.0010
−0.0015
0.0
0.2
0.4
0.6
0.8
1.0
Viscosity Ratio m
Δ=0.0
Δ=0.2
Δ=0.4
Δ=0.6

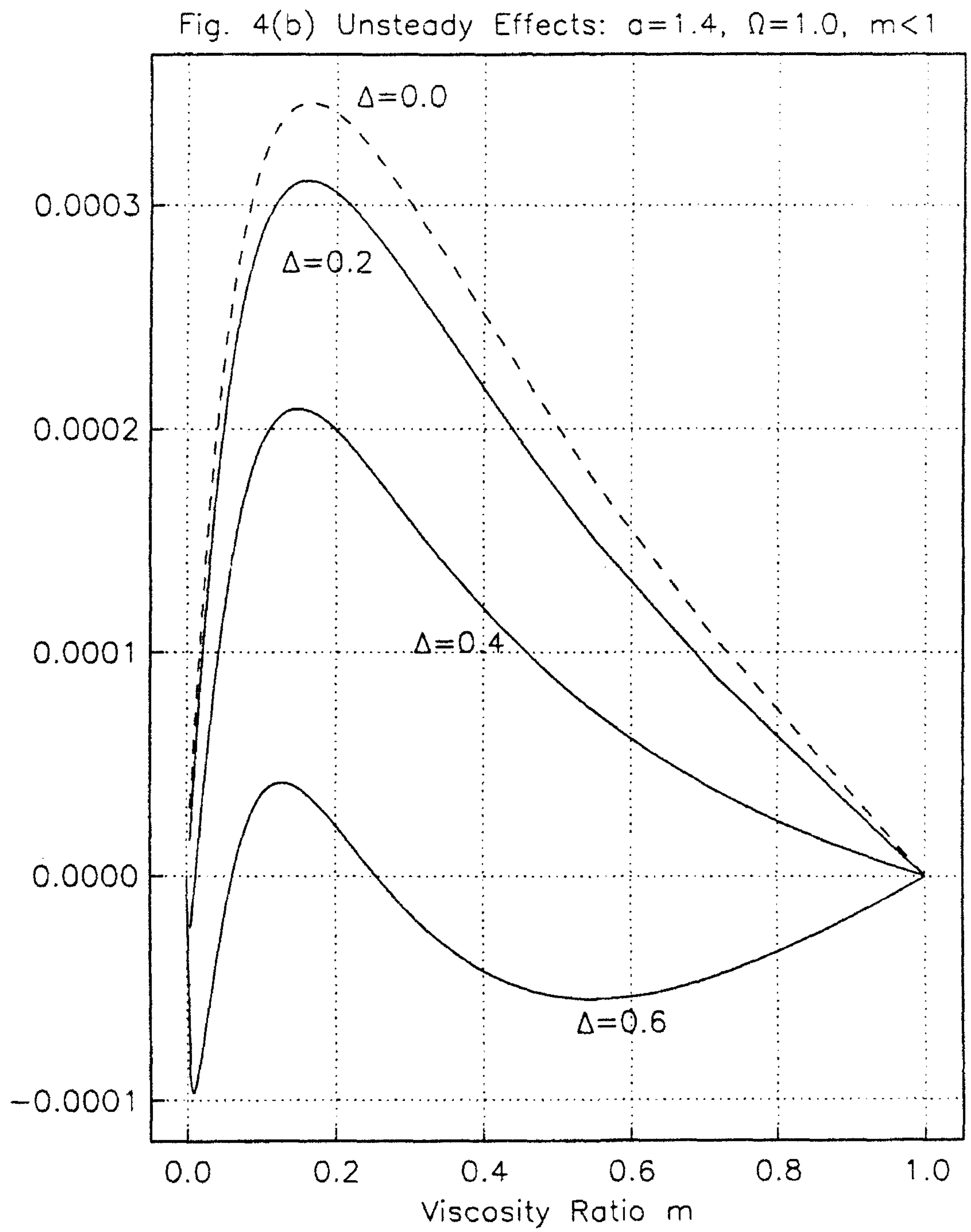


Fig. 4(b) Unsteady Effects: a=1.4, Ω=1.0, m<1
Δ=0.0
Δ=0.2
Δ=0.4
Δ=0.6
0.0003
0.0002
0.0001
0.0000
−0.0001
ϑ
0.0
0.2
0.4
0.6
0.8
1.0
Viscosity Ratio m

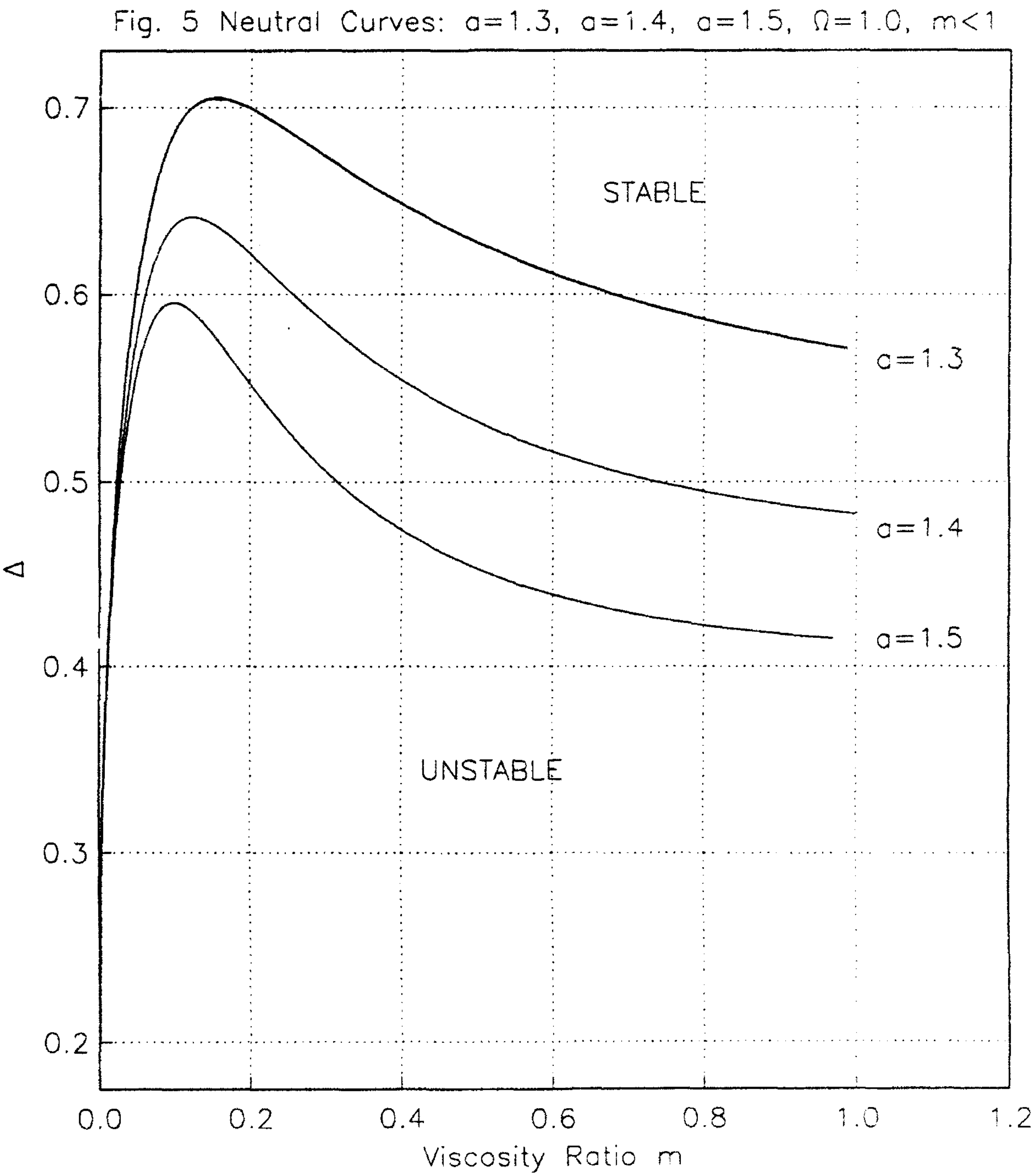


Fig. 5 Neutral Curves: a=1.3, a=1.4, a=1.5, Ω=1.0, m<1
STABLE
UNSTABLE
a=1.3
a=1.4
a=1.5
0.7
0.6
0.5
0.4
0.3
0.2
Δ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Viscosity Ratio m

Fig. 6(a) Frequency Effects: a=1.4, m>1

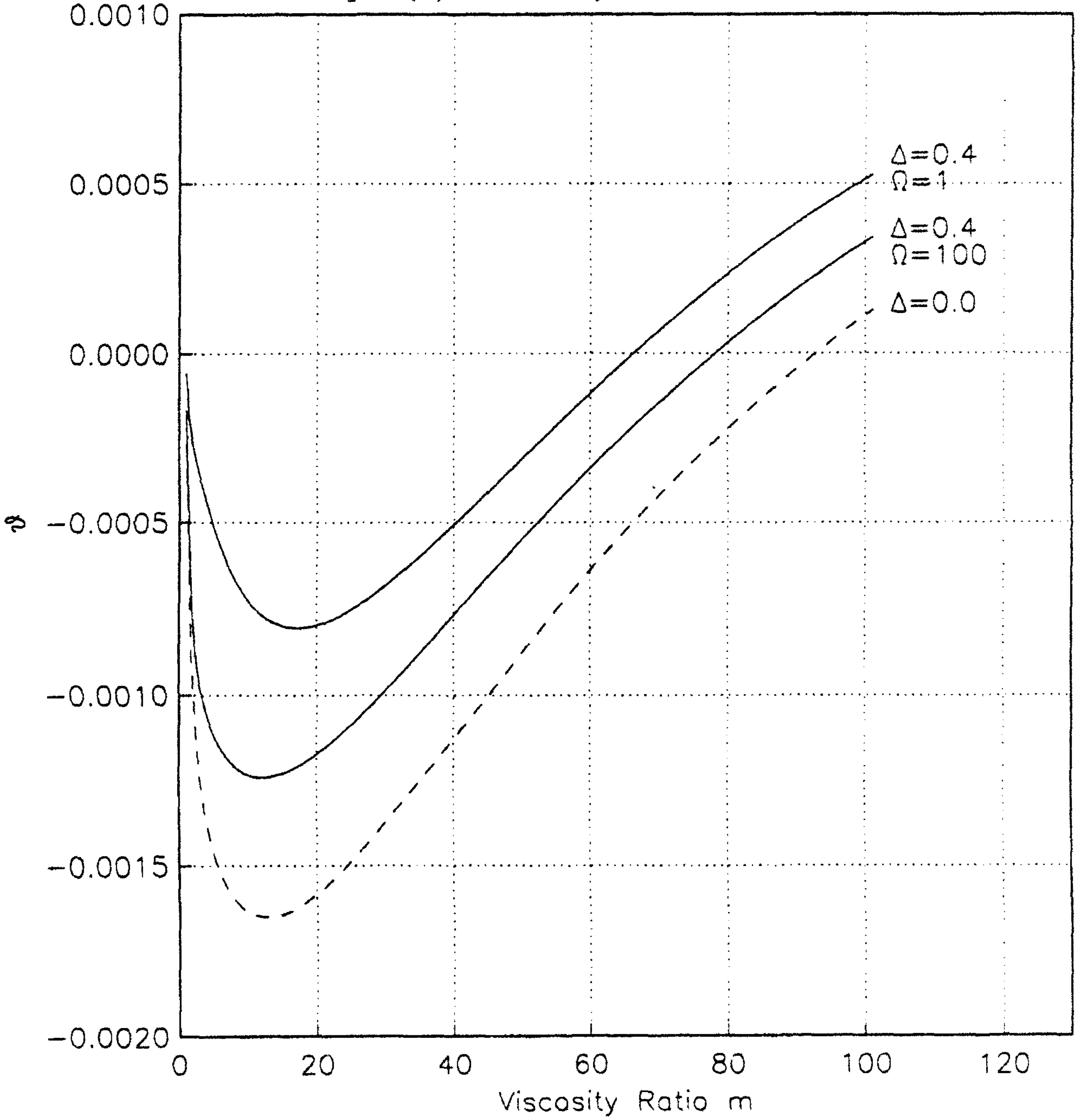
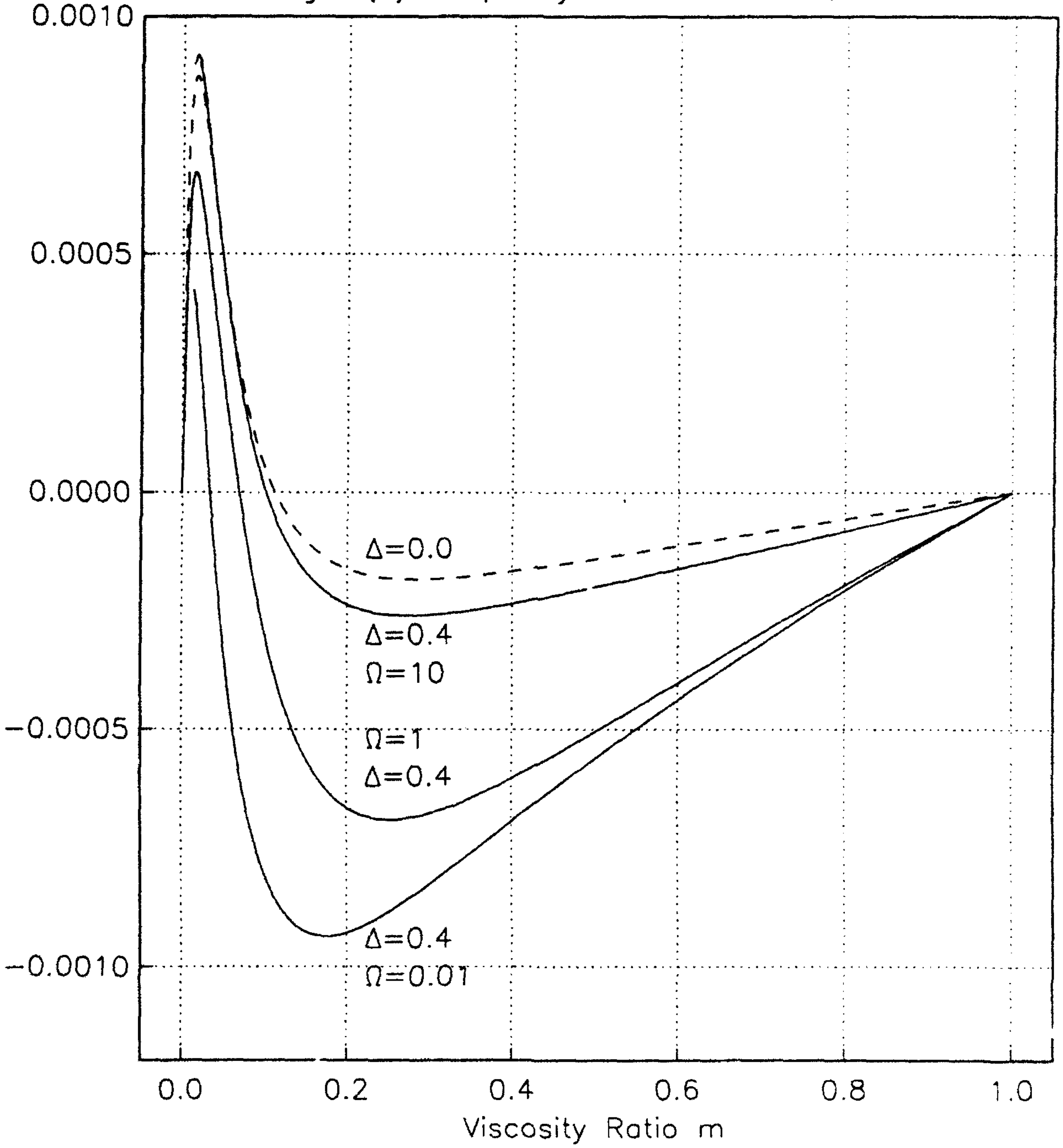


Fig. 6(b) Frequency Effects: a=2.25, m<1



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We investigate the stability of two phase Couette flow of different liquids bounded between plane parallel plates. One of the plates has a time dependent velocity in its own plane, which is composed of a constant steady part and a time harmonic component. In the absence of time harmonic modulations the flow can be unstable to an interfacial instability if the viscosities are different and the more viscous fluid occupies the thinner of the two layers. Using Floquet theory, we show analytically in the limit of long waves, that time periodic modulations in the basic flow can have a significant influence on flow stability. In particular, flows which are otherwise unstable for extensive ranges of viscosity ratios, can be stabilized completely by the inclusion of background modulations, a finding that can have useful consequences in many practical applications.

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